Chapter

5.3

Hyperbola

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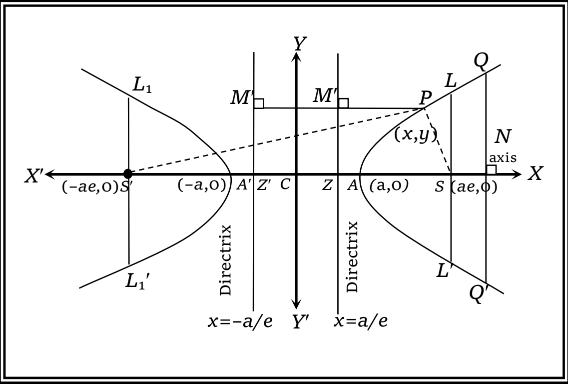


Apollonius writes Conics in which he introduces the terms "parabola", " ellipse" and "hyperbola".

De Beaune writes Notes brieves which contains the many results on "Cartesian geometry", in particular giving the now familiar equations for hyperbolas, parabolas and ellipses.

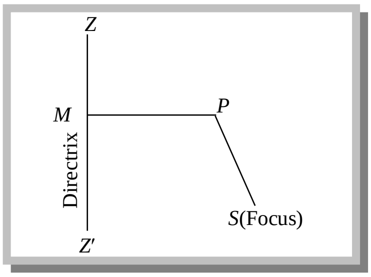
The hyperbola is also useful for describing the path of an alpha particle in the electric field of the nucleus of an atom.

Hyperbola has its application in the field of Ballistics. Suppose a gun is fired. If the sound reaches two listening posts, situated at two foci of the hyperbola at different times, from the time difference, the distance between the two listening posts (two foci) can be calculated.

5.3 Hyperbola

5.3.1 Definition

A hyperbola is the locus of a point in a plane which moves in the plane in such a way that the ratio of its distance from a fixed point in the same plane to its distance from a fixed line is always constant which is always greater than unity.



Fixed point is called focus, fixed straight line is called directrix and the constant ratio is called eccentricity of the hyperbola. Eccentricity is denoted by e and $e > 1$.

A hyperbola is the particular case of the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

When, $abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$ *i.e.,* $\Delta \neq 0$ and $h^2 > ab$.

Let $S(h, k)$ is the focus, directrix is the line $ax + by + c = 0$ and the eccentricity is e. Let $P(x_1, y_1)$ be a point which moves such that $SP = e.PM$

$$\Rightarrow \sqrt{(x_1 - h)^2 + (y_1 - k)^2} = e.\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}$$

$$\Rightarrow (a^2 + b^2)[(x_1 - h)^2 + (y_1 - k)^2] = e^2(ax_1 + by_1 + c)^2$$

Hence, locus of (x_1, y_1) is given by $(a^2 + b^2)[(x - h)^2 + (y - k)^2] = e^2(ax + by + c)^2$

Which is a second degree equation to represent a hyperbola ($e > 1$).

Example: 1 The equation of the conic with focus at (1, – 1), directrix along $x - y + 1 = 0$ and with eccentricity $\sqrt{2}$ is

[EAMCET 1994; DCE 1998]

(a) $x^2 - y^2 = 1$ (b) $xy = 1$ (c) $2xy - 4x + 4y + 1 = 0$ (d) $2xy + 4x - 4y - 1 = 0$

Solution: (c) Here, focus (S) = (1, –1), eccentricity (e)= $\sqrt{2}$

From definition, $SP = e\,PM$

$$\sqrt{(x - 1)^2 + (y + 1)^2} = \frac{\sqrt{2}.(x - y + 1)}{\sqrt{1^2 + 1^2}}$$

$\Rightarrow (x - 1)^2 + (y + 1)^2 = (x - y + 1)^2 \Rightarrow 2xy - 4x + 4y + 1 = 0$, which is the required equation of conic (Rectangular hyperbola)

Example: 2 The centre of the hyperbola $9x^2 - 36x - 16y^2 + 96y - 252 = 0$ is **[Karnataka CET 1993]**

(a) (2, 3) (b) (– 2, – 3) (c) (–2, 3) (d) (2, – 3)

Solution: (a) Here $a = 9$, $b = -16$, $h = 0$, $g = -18$, $f = 48$, $c = -252$

Centre of hyperbola = $\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right) = \left(\frac{(0)(48) - (-16)(-18)}{(9)(-16) - 0}, \frac{(-18)(0) - (9)(48)}{(9)(-16) - 0}\right)$ = (2, 3)

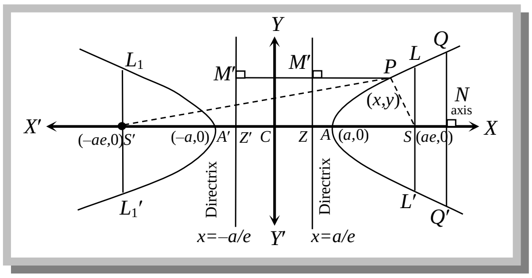
5.3.2 Standard equation of the Hyperbola

Let S be the focus, ZM be the directrix and e be the eccentricity of the hyperbola, then by definition,

$$\Rightarrow \frac{SP}{PM} = e \Rightarrow (SP)^2 = e^2 (PM)^2$$

$$\Rightarrow (x - a.e)^2 + (y - 0)^2 = e^2 \left(x - \frac{a}{e} \right)^2$$

$$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1 \Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{, where } b^2 = a^2(e^2 - 1)$$



This is the standard equation of the hyperbola.

Some terms related to hyperbola : Let the equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(1) **Centre :** All chords passing through C are bisected at C. Here $C(0,0)$

(2) **Vertex:** The point A and A' where the curve meets the line joining the foci S and S' are called vertices of hyperbola. The co-ordinates of A and A' are $(a, 0)$ and $(-a, 0)$ respectively.

(3) **Transverse and conjugate axes :** The straight line joining the vertices A and A′ is called transverse axis of the hyperbola. The straight line perpendicular to the transverse axis and passing through the centre is called conjugate axis.

Here, transverse axis = $AA' = 2a$

Conjugate axis = $BB' = 2b$

(4) **Eccentricity :** For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

We have $b^2 = a^2(e^2 - 1)$, $e = \sqrt{1 + \left(\frac{2b}{2a}\right)^2} = \sqrt{1 + \left(\frac{\text{Conjugate axis}}{\text{Transverse axis}}\right)^2}$

(5) **Double ordinates :** If Q be a point on the hyperbola, QN perpendicular to the axis of the hyperbola and produced to meet the curve again at Q'. Then QQ' is called a double ordinate at Q.

If abscissa of Q is h, then co-ordinates of Q and Q' are $\left(h, \frac{b}{a}\sqrt{h^2 - a^2}\right)$ and $\left(h, -\frac{b}{a}\sqrt{h^2 - a^2}\right)$ respectively.

(6) **Latus-rectum :** The chord of the hyperbola which passes through the focus and is perpendicular to its transverse axis is called latus-rectum.

Length of latus-rectum $LL' = L_1L_1' = \frac{2b^2}{a} = 2a(e^2 - 1)$ and end points of latus-rectum $L\left(ae, \frac{b^2}{a}\right)$; $L'\left(ae, \frac{-b^2}{a}\right)$; $L_1\left(-ae, \frac{b^2}{a}\right)$; $L_1'\left(-ae, -\frac{b^2}{a}\right)$ respectively.

(7) **Foci and directrices:** The points $S(ae, 0)$ and $S'(-ae, 0)$ are the foci of the hyperbola and ZM and $Z'M'$ are two directrices of the hyperbola and their equations are $x = \frac{a}{e}$ and $x = -\frac{a}{e}$ respectively.

Distance between foci $SS' = 2ae$ and distance between directrices $ZZ' = 2a/e$.

(8) **Focal chord :** A chord of the hyperbola passing through its focus is called a focal chord.

(9) **Focal distance :** The difference of any point on the hyperbola from the focus is called the focal distance of the point.

From the figure, $SP = ePM = e\left(x_1 - \frac{a}{e}\right) = ex_1 - a$, $S'P = ePM' = e\left(x_1 + \frac{a}{e}\right) = ex_1 + a$

The difference of the focal distance of a point on the hyperbola is constant and is equal to the length of transverse axis.

$|S'P - SP| = 2a = AA' =$ Transverse axis

Example: 3 The eccentricity of the hyperbola which passes through (3, 0) and $(3\sqrt{2}, 2)$ is **[UPSEAT 2000]**

(a) $\sqrt{(13)}$ (b) $\frac{\sqrt{13}}{3}$ (c) $\sqrt{\frac{13}{4}}$ (d) None of these

Solution: (b) Let equation of hyperbola is $x^2/a^2 - y^2/b^2 = 1$. Point (3, 0) lies on hyperbola

So, $\frac{(3)^2}{a^2} - \frac{0}{b^2} = 1$ or $\frac{9}{a^2} = 1$ or $a^2 = 9$ and point $(3\sqrt{2}, 2)$ also lies on hyperbola. So, $\frac{3(\sqrt{2})^2}{a^2} - \frac{(2)^2}{b^2} = 1$

Put $a^2 = 9$ we get, $\frac{18}{9} - \frac{4}{b^2} = 1$ or $2 - \frac{4}{b^2} = 1$ or $-\frac{4}{b^2} = 1 - 2$ or $\frac{4}{b^2} = 1$ or $b^2 = 4$

We know that $b^2 = a^2(e^2 - 1)$. Putting values of a^2 and b^2

$4 = 9(e^2 - 1)$ or $e^2 - 1 = \frac{4}{9}$ or $e^2 = 1 + \frac{4}{9}$ or $e = \sqrt{(1 + 4/9)}$ or $e = \sqrt{(13)/9} = \frac{\sqrt{13}}{3}$.

Example: 4 The foci of the hyperbola $9x^2 - 16y^2 = 144$ are **[MP PET 2001]**

(a) $(\pm 4, 0)$ (b) $(0, \pm 4)$ (c) $(\pm 5, 0)$ (d) $(0, \pm 5)$

Solution: (c) The equation of hyperbola is $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

Now, $b^2 = a^2(e^2 - 1) \Rightarrow 9 = 16(e^2 - 1) \Rightarrow e = \frac{5}{4}$. Hence foci are $(\pm ae, 0) = \left(\pm 4 . \frac{5}{4}, 0\right)$ *i.e.*, $(\pm 5, 0)$

Example: 5 If the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ coincide, then the value of b^2 is

[MNR 1992; UPSEAT 2001; AIEEE 2003]

(a) 1 (b) 5 (c) 7 (d) 9

Solution: (c) For hyperbola, $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$

$A = \sqrt{\frac{144}{25}}, B = \sqrt{\frac{81}{25}}, e_1 = \sqrt{1 + \frac{B^2}{A^2}} = \sqrt{1 + \frac{81}{144}} = \sqrt{\frac{225}{144}} = \frac{5}{4}$

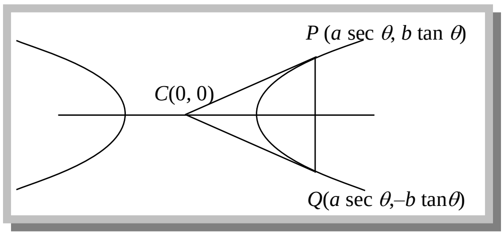
Therefore foci = $(\pm ae_1, 0) = \left(\pm \frac{12}{5} . \frac{5}{4}, 0\right) = (\pm 3, 0)$. Therefore foci of ellipse *i.e.*, $(\pm 4e, 0) = (\pm 3, 0)$ (For ellipse $a = 4$)

$\Rightarrow e = \frac{3}{4}$, Hence $b^2 = 16\left(1 - \frac{9}{16}\right) = 7$.

Example: 6 If PQ is a double ordinate of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ such that CPQ is an equilateral triangle, C being the centre of the hyperbola. Then the eccentricity e of the hyperbola satisfies **[EAMCET 1999]**

(a) $1 < e < 2/\sqrt{3}$ (b) $e = 2/\sqrt{3}$ (c) $e = \sqrt{3}/2$ (d) $e > 2/\sqrt{3}$

Solution: (d) Let $P(a\sec\theta, b\tan\theta)$; $Q(a\sec\theta, -b\tan\theta)$ be end points of double ordinates and $C(0,0)$ is the centre of the hyperbola



Now $PQ = 2b\tan\theta$; $CQ = CP = \sqrt{a^2\sec^2\theta + b^2\tan^2\theta}$

Since $CQ = CP = PQ$, $\therefore$ $4b^2\tan^2\theta = a^2\sec^2\theta + b^2\tan^2\theta$

$\Rightarrow 3b^2\tan^2\theta = a^2\sec^2\theta \Rightarrow 3b^2\sin^2\theta = a^2$

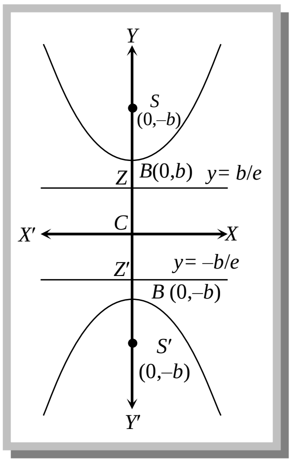
$\Rightarrow 3a^2(e^2 - 1)\sin^2\theta = a^2 \Rightarrow 3(e^2 - 1)\sin^2\theta = 1$

$\Rightarrow \frac{1}{3(e^2 - 1)} = \sin^2\theta < 1$ $(\because \sin^2\theta < 1)$

$\Rightarrow \frac{1}{e^2 - 1} < 3 \Rightarrow e^2 - 1 > \frac{1}{3} \Rightarrow e^2 > \frac{4}{3} \Rightarrow e > \frac{2}{\sqrt{3}}$

5.3.3 Conjugate Hyperbola

The hyperbola whose transverse and conjugate axis are respectively the conjugate and transverse axis of a given hyperbola is called conjugate hyperbola of the given hyperbola.



Fundamentals \ Hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ or $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$
Centre	(0, 0)	(0, 0)
Length of transverse axis	$2a$	$2b$
Length of conjugate axis	$2b$	$2a$
Foci	$(\pm ae, 0)$	$(0, \pm be)$
Equation of directrices	$x = \pm a/e$	$y = \pm b/e$
Eccentricity	$e=\sqrt{\left(\frac{a^2+b^2}{a^2}\right)}$	$e=\sqrt{\left(\frac{a^2+b^2}{b^2}\right)}$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Parametric co-ordinates	$(a\sec\phi, b\tan\phi)$, $0\le\phi<2\pi$	$(b\sec\phi, a\tan\phi)$, $0\le\phi<2\pi$
Focal radii	$SP = ex_1 - a$ & $S'P = ex_1 + a$	$SP = ey_1 - b$ & $S'P = ey_1 + b$
Difference of focal radii $(S'P - SP)$	$2a$	$2b$
Tangents at the vertices	$x=-a, x=a$	$y=-b, y=b$
Equation of the transverse axis	$y=0$	$x=0$
Equation of the conjugate axis	$x=0$	$y=0$

Note : ❑ If e and e' are the eccentricities of a hyperbola and its conjugate, then $\frac{1}{e^2}+\frac{1}{e'^2}=1$.

❑ The foci of a hyperbola and its conjugate are concyclic.

Example: 7 The eccentricity of the conjugate hyperbola of the hyperbola $x^2-3y^2=1$, is **[MP PET 1999]**

(a) 2 (b) $\frac{2}{\sqrt{3}}$ (c) 4 (d) $\frac{4}{3}$

Solution: (a) The given hyperbola is $\frac{x^2}{1}-\frac{y^2}{1/3}=1$. Here $a^2=1$ and $b^2=\frac{1}{3}$

Since $b^2=a^2(e^2-1) \Rightarrow \frac{1}{3}=1(e^2-1) \Rightarrow e^2=\frac{4}{3} \Rightarrow e=\frac{2}{\sqrt{3}}$

If e' is the eccentricity of the conjugate hyperbola, then $\frac{1}{e^2}+\frac{1}{e'^2}=1 \Rightarrow \frac{1}{e'^2}=1-\frac{1}{e^2}=1-\frac{3}{4}=\frac{1}{4} \Rightarrow e'=2$.

5.3.4 Special form of Hyperbola

If the centre of hyperbola is (h, k) and axes are parallel to the co-ordinate axes, then its equation is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$. By shifting the origin at (h, k) without rotating the co-ordinate axes, the above equation reduces to $\frac{X^2}{a^2}-\frac{Y^2}{b^2}=1$, where $x=X+h, y=Y+k$.

Example: 8 The equation of the hyperbola whose foci are (6, 4) and (– 4, 4) and eccentricity 2 is given by **[MP PET 1993]**

(a) $12x^2-4y^2-24x+32y-127=0$ (b) $12x^2+4y^2+24x-32y-127=0$

(c) $12x^2 - 4y^2 - 24x - 32y + 127 = 0$ (d) $12x^2 - 4y^2 + 24x + 32y + 127 = 0$

Solution: (a) Foci are (6, 4) and (– 4, 4) and $e = 2$.

$\therefore$ Centre is $\left(\frac{6-4}{2}, \frac{4+4}{2}\right) = (1, 4)$

So, $ae + 1 = 6 \Rightarrow ae = 5 \Rightarrow a = \frac{5}{2}$ and $b = \frac{5}{2}\sqrt{3}$

Hence, the required equation is $\frac{(x-1)^2}{25/4} - \frac{(y-4)^2}{(75/4)} = 1$ or $12x^2 - 4y^2 - 24x + 32y - 127 = 0$

Example: 9 The equations of the directrices of the conic $x^2 + 2x - y^2 + 5 = 0$ are

(a) $x = \pm 1$ (b) $y = \pm 2$ (c) $y = \pm\sqrt{2}$ (d) $x = \pm\sqrt{3}$

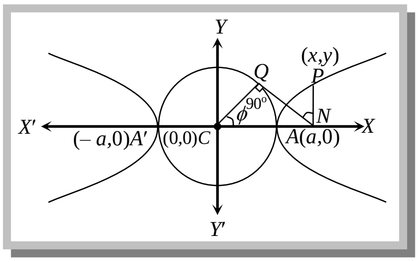
Solution: (c) $(x+1)^2 - y^2 - 1 + 5 = 0 \Rightarrow -\frac{(x+1)^2}{4} + \frac{y^2}{4} = 1$

Equation of directrices of $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ are $y = \pm\frac{b}{e}$

Here $b = 2$, $e = \sqrt{1+1} = \sqrt{2}$. Hence, $y = \pm\frac{2}{\sqrt{2}} \Rightarrow y = \pm\sqrt{2}$.

5.3.5 Auxiliary circle of Hyperbola

Let $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be the hyperbola with centre C and transverse axis $A'A$. Therefore circle drawn with centre C and segment $A'A$ as a diameter is called auxiliary circle of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



$\therefore$ Equation of the auxiliary circle is $x^2 + y^2 = a^2$

Let $\angle QCN = \phi$

Here P and Q are the corresponding points on the hyperbola and the auxiliary circle $(0 \le \phi < 2\pi)$

(1) **Parametric equations of hyperbola :** The equations $x = a\sec\phi$ and $y = b\tan\phi$ are known as the parametric equations of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. This $(a\sec\phi, b\tan\phi)$ lies on the hyperbola for all values of ϕ.

Position of points Q on auxiliary circle and the corresponding point P which describes the hyperbola and $0 \le \phi < 2\pi$

ϕ varies from	$Q(a\cos\varphi, a\sin\varphi)$	$P(a\sec\varphi, b\tan\varphi)$
0 to $\frac{\pi}{2}$	I	I
$\frac{\pi}{2}$ to π	II	III
π to $\frac{3\pi}{2}$	III	II
$\frac{3\pi}{2}$ to 2π	IV	IV

Note : ❑ The equations $x = a\cosh\theta$ and $y = b\sin h\theta$ are also known as the parametric equations of the hyperbola and the co-ordinates of any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are expressible as $(a\cos h\,\theta,\ b\sin h\,\theta)$, where $\cos h\,\theta = \frac{e^{\theta} + e^{-\theta}}{2}$ and $\sin h\,\theta = \frac{e^{\theta} - e^{-\theta}}{2}$.

Example: 10 The distance between the directrices of the hyperbola $x = 8\ \sec\theta$, $y = 8\ \tan\theta$ is **[Karnataka CET 2003]**

(a) $16\sqrt{2}$ (b) $\sqrt{2}$ (c) $8\sqrt{2}$ (d) $4, \sqrt{2}$

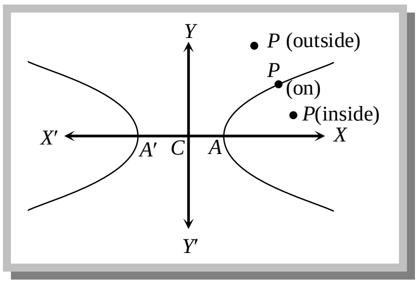
Solution: (c) Equation of hyperbola is $x = 8\sec\theta, y = 8\tan\theta \Rightarrow \frac{x}{8} = \sec\theta$, $\frac{y}{8} = \tan\theta$

$\therefore \sec^2\theta - \tan^2\theta = 1 \Rightarrow \frac{x^2}{8^2} - \frac{y^2}{8^2} = 1$

Here $a = 8, b = 8$. Now $e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{8^2}{8^2}} = \sqrt{2}$

$\therefore$ Distance between directrices $= \frac{2a}{e} = \frac{2 \times 8}{\sqrt{2}} = 8\sqrt{2}$.

5.3.6 Position of a point with respect to a Hyperbola



Let the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Then $P(x_1, y_1)$ will lie inside, on or outside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ according as $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$ is positive, zero or negative.

Example: 11 The number of tangents to the hyperbola $\frac{x^2}{4} - \frac{y^2}{3} = 1$ through (4, 1) is **[AMU 1998]**

(a) 1 (b) 2 (c) 0 (d) 3

Solution: (c) Since the point (4, 1) lies inside the hyperbola $\left[\because \frac{16}{4} - \frac{1}{3} - 1 > 0\right]$; $\therefore$ Number of tangents through (4, 1) is 0.

5.3.7 Intersection of a Line and a Hyperbola

The straight line $y = mx + c$ will cut the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in two points may be real, coincident or imaginary according as $c^2 >, =, < a^2m^2 - b^2$.

Condition of tangency : If straight line $y = mx + c$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $c^2 = a^2m^2 - b^2$.

5.3.8 Equations of Tangent in Different forms

(1) **Point form :** The equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

(2) **Parametric form :** The equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(a\sec\phi, b\tan\phi)$ is

$$\frac{x}{a}\sec\phi - \frac{y}{b}\tan\phi = 1$$

(3) **Slope form :** The equations of tangents of slope m to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are $y=mx\pm\sqrt{a^2m^2-b^2}$ and the co-ordinates of points of contacts are $\left(\pm\frac{a^2m}{\sqrt{a^2m^2-b^2}},\pm\frac{b^2}{\sqrt{a^2m^2-b^2}}\right)$.

Note : ❑ If the straight line $lx+my+n=0$ touches the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then $a^2l^2-b^2m^2=n^2$.

❑ If the straight line $x\cos\alpha+y\sin\alpha=p$ touches the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$,

then $a^2\cos^2\alpha-b^2\sin^2\alpha=p^2$

❑ Two tangents can be drawn from an outside point to a hyperbola.

Important Tips

☞ *For hyperbola* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ *and* $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$, *the equation of common tangent is* $y=\pm x\pm\sqrt{a^2-b^2}$, *points of contacts are* $\left(\pm\frac{a^2}{\sqrt{a^2-b^2}};\pm\frac{b^2}{\sqrt{a^2-b^2}}\right)$ *and length of common tangent is* $\sqrt{2}.\frac{(a^2+b^2)}{\sqrt{a^2-b^2}}$.

☞ *If the line* $y=mx\pm\sqrt{a^2m^2-b^2}$ *touches the hyperbola* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ *at the point* $(a\sec\theta, b\tan\theta)$, *then* $\theta=\sin^{-1}\left(\frac{b}{am}\right)$.

Example: 12 The value of m for which $y=mx+6$ is a tangent to the hyperbola $\frac{x^2}{100}-\frac{y^2}{49}=1$, is **[Karnataka CET 1993]**

(a) $\sqrt{\frac{17}{20}}$ (b) $\sqrt{\frac{20}{17}}$ (c) $\sqrt{\frac{3}{20}}$ (d) $\sqrt{\frac{20}{3}}$

Solution: (a) For condition of tangency, $c^2=a^2m^2-b^2$. Here $c=6$, $a=10, b=7$

Then, $(6)^2=(10)^2.m^2-(7)^2$

$36=100m^2-49 \Rightarrow 100m^2=85 \Rightarrow m^2=\frac{17}{20} \Rightarrow m=\sqrt{\frac{17}{20}}$

Example: 13 If m_1 and m_2 are the slopes of the tangents to the hyperbola $\frac{x^2}{25}-\frac{y^2}{16}=1$ which pass through the point (6, 2), then

(a) $m_1+m_2=\frac{24}{11}$ (b) $m_1m_2=\frac{20}{11}$ (c) $m_1+m_2=\frac{48}{11}$ (d) $m_1m_2=\frac{11}{20}$

Solution: (a, b) The line through (6, 2) is $y-2=m(x-6) \Rightarrow y=mx+2-6m$

Now, from condition of tangency $(2-6m)^2=25m^2-16$

$\Rightarrow 36m^2+4-24m-25m^2+16=0 \Rightarrow 11m^2-24m+20=0$

Obviously, its roots are m_1 and m_2, therefore $m_1+m_2=\frac{24}{11}$ and $m_1m_2=\frac{20}{11}$

Example: 14 The points of contact of the line $y=x-1$ with $3x^2-4y^2=12$ is **[BIT Ranchi 1996]**

(a) (4, 3) (b) (3, 4) (c) (4, – 3) (d) None of these

Solution: (a) The equation of line and hyperbola are $y=x-1$ .....(i) and $3x^2-4y^2=12$ .....(ii)

From (i) and (ii), we get $3x^2-4(x-1)^2=12$

$\Rightarrow 3x^2-4(x^2-2x+1)=12$ or $x^2-8x+16=0 \Rightarrow x=4$

From (i), $y=3$ so points of contact is (4, 3)

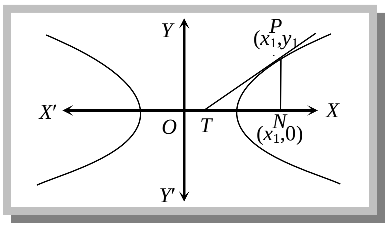
Trick : Points of contact are $\left(\pm\frac{a^2m}{\sqrt{a^2m^2-b^2}},\pm\frac{b^2}{\sqrt{a^2m^2-b^2}}\right)$.

Here $a^2=4$, $b^2=3$ and $m=1$. So the required points of contact is (4, 3).

Example: 15 P is a point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, N$ is the foot of the perpendicular from P on the transverse axis. The tangent to the hyperbola at P meets the transverse axis at T. If O is the centre of the hyperbola, then $OT.ON$ is equal to

(a) e^2 (b) a^2 (c) b^2 (d) $\frac{b^2}{a^2}$

Solution: (b) Let $P(x_1,y_1)$ be a point on the hyperbola. Then the co-ordinates of N are $(x_1,0)$.



The equation of the tangent at (x_1,y_1) is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$

This meets x-axis at $T\left(\frac{a^2}{x_1},0\right)$; $\therefore$ $OT.ON=\frac{a^2}{x_1}\times x_1=a^2$

Example: 16 If the tangent at the point $(2\sec\phi, 3\tan\phi)$ on the hyperbola $\frac{x^2}{4}-\frac{y^2}{9}=1$ is parallel to $3x-y+4=0$, then the value of ϕ is

(a) 45^o (b) 60^o (c) 30^o (d) 75^o

Solution: (c) Here $x=2\sec\phi$ and $y=3\tan\phi$

Differentiating *w.r.t.* ϕ

$\frac{dx}{d\phi}=2\sec\phi\tan\phi$ and $\frac{dy}{d\phi}=3\sec^2\phi$

$\therefore$ Gradient of tangent $\frac{dy}{dx}=\frac{dy/d\phi}{dx/d\phi}=\frac{3\sec^2\phi}{2\sec\phi\tan\phi}$; $\therefore$ $\frac{dy}{dx}=\frac{3}{2}\text{cosec}\,\phi$ .....(i)

But tangent is parallel to $3x-y+4=0$; $\therefore$ Gradient $m=3$ ......(ii)

From (i) and (ii), $\frac{3}{2}\text{cosec}\,\phi=3 \Rightarrow$ $\text{cosec}\,\phi=2$, $\therefore$ $\phi=30^o$

Example: 17 The slopes of the common tangents to the hyperbola $\frac{x^2}{9}-\frac{y^2}{16}=1$ and $\frac{y^2}{9}-\frac{x^2}{16}=1$ are **[Roorkee 1997]**

(a) – 2, 2 (b) – 1, 1 (c) 1, 2 (d) 2, 1

Solution: (b) Given hyperbola are $\frac{x^2}{9}-\frac{y^2}{16}=1$ .....(i) and $\frac{y^2}{9}-\frac{x^2}{16}=1$ .....(ii)

Any tangent to (i) having slope m is $y=mx\pm\sqrt{9m^2-16}$ .....(iii)

Putting in (ii), we get, $16[mx\pm\sqrt{9m^2-16}]^2-9x^2=144$

$\Rightarrow (16m^2-9)x^2\pm 32m(\sqrt{9m^2-16})x+144m^2-256-144=0$

$\Rightarrow (16m^2-9)x^2\pm 32m(\sqrt{9m^2-16})x+(144m^2-400)=0$ .....(iv)

If (iii) is a tangent to (ii), then the roots of (iv) are real and equal.

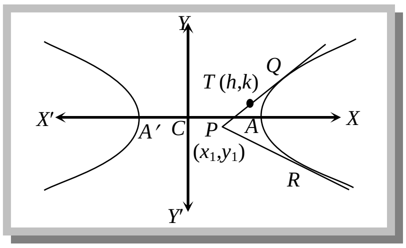
$\therefore$ Discriminant = 0; $32\times 32m^2(9m^2-16) = 4(16m^2-9)(144m^2-400) = 64(16m^2-9)(9m^2-25)$

$\Rightarrow 16m^2(9m^2-16)=(16m^2-9)(9m^2-25) \Rightarrow 144m^4-256m^2=144m^4-481m^2+225$

$\Rightarrow 225m^2=225 \Rightarrow m^2=1 \Rightarrow m=\pm 1$

5.3.9 Equation of Pair of Tangents

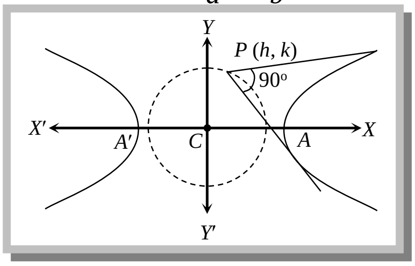
If $P(x_1,y_1)$ be any point outside the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ then a pair of tangents PQ, PR can be drawn to it from P.



The equation of pair of tangents PQ and PR is $SS_1=T^2$

where, $S=\frac{x^2}{a^2}-\frac{y^2}{b^2}-1$, $S_1=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$, $T=\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$

Director circle : The director circle is the locus of points from which perpendicular tangents are drawn to the given hyperbola. The equation of the director circle of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $x^2+y^2=a^2-b^2$



Example: 18 The locus of the point of intersection of tangents to the hyperbola $4x^2-9y^2=36$ which meet at a constant angle $\pi/4$, is

(a) $(x^2+y^2-5)^2=4(9y^2-4x^2+36)$ (b) $(x^2+y^2-5)=4(9y^2-4x^2+36)$

(c) $4(x^2+y^2-5)^2=(9y^2-4x^2+36)$ (d) None of these

Solution: (a) Let the point of intersection of tangents be $P(x_1,y_1)$. Then the equation of pair of tangents from $P(x_1,y_1)$ to the given hyperbola is $(4x^2-9y^2-36)(4x_1^2-9y_1^2-36)=[4x_1x-9y_1y-36]^2$ ......(i)

From $SS_1=T^2$ or $x^2(y_1^2+4)+2x_1y_1xy+y^2(x_1^2-9)+.....=0$ .....(ii)

Since angle between the tangents is $\pi/4$.

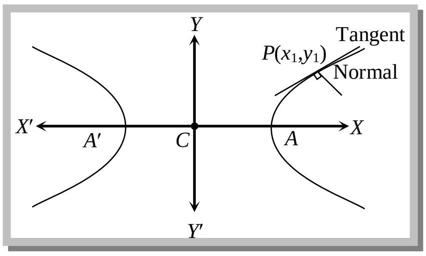
$\therefore\ \tan(\pi/4)=\frac{2\sqrt{[x_1^2y_1^2-(y_1^2+4)(x_1^2-9)]}}{y_1^2+4+x_1^2-9}$. Hence locus of $P(x_1,y_1)$ is $(x^2+y^2-5)^2=4(9y^2-4x^2+36)$.

5.3.10 Equations of Normal in Different forms

(1) **Point form :** The equation of normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at (x_1,y_1) is $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2$.

(2) **Parametric form:** The equation of normal at $(a\sec\theta, b\tan\theta)$ to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$ax\cos\theta+by\cot\theta=a^2+b^2$$



(3) **Slope form:** The equation of the normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ in terms of the slope m of the normal is $y=mx\mp\frac{m(a^2+b^2)}{\sqrt{a^2-b^2m^2}}$

(4) **Condition for normality :** If $y=mx+c$ is the normal of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

then $c=\mp\frac{m(a^2+b^2)}{\sqrt{a^2-m^2b^2}}$ or $c^2=\frac{m^2(a^2+b^2)^2}{(a^2-m^2b^2)}$, which is condition of normality.

(5) **Points of contact :** Co-ordinates of points of contact are $\left(\pm\frac{a^2}{\sqrt{a^2-b^2m^2}},\mp\frac{mb^2}{\sqrt{a^2-b^2m^2}}\right)$

Note : ❑ If the line $lx + my + n = 0$ will be normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $\frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2+b^2)^2}{n^2}$.

Important Tip

☞ *In general, four normals can be drawn to a hyperbola from any point and if* $\alpha, \beta, \gamma, \delta$ *be the eccentric angles of these four co-normal points, then* $\alpha + \beta + \gamma + \delta$ *is an odd multiple of* π.

☞ *If* α, β, γ *are the eccentric angles of three points on the hyperbola.* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, *the normals at which are concurrent, then,* $\sin(\alpha+\beta) + \sin(\beta+\gamma) + \sin(\gamma+\alpha) = 0$

☞ *If the normal at P meets the transverse axis in G, then* $SG = e.SP$. *Also the tangent and normal bisect the angle between the focal distances of P.*

☞ *The feet of the normals to* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ *from* (h,k) *lie on* $a^2y(x-h) + b^2x(y-k) = 0$.

Example: 19 The equation of the normal to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ at the point $(8, 3\sqrt{3})$ is **[MP PET 1996]**

(a) $\sqrt{3}x + 2y = 25$ (b) $x + y = 25$ (c) $y + 2x = 25$ (d) $2x + \sqrt{3}y = 25$

Solution: (d) From $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$

Here $a^2 = 16$, $b^2 = 9$ and $(x_1, y_1) = (8, 3\sqrt{3})$

$\Rightarrow \frac{16x}{8} + \frac{9y}{3\sqrt{3}} = 16 + 9$ *i.e.,* $2x + \sqrt{3}y = 25$.

Example: 20 If the normal at 'ϕ' on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets transverse axis at *G*, then $AG.A'G =$

(Where *A* and A' are the vertices of the hyperbola)

(a) $a^2(e^4\sec^2\phi - 1)$ (b) $(a^2e^4\sec^2\phi - 1)$ (c) $a^2(1 - e^4\sec^2\phi)$ (d) None of these

Solution: (a) The equation of normal at $(a\sec\phi, b\tan\phi)$ to the given hyperbola is $ax\cos\phi + by\cot\phi = (a^2 + b^2)$

This meets the transverse axis *i.e., x*-axis at *G*. So the co-ordinates of *G* are $\left(\left(\frac{a^2+b^2}{a}\right)\sec\phi, 0\right)$ and the co-ordinates of the vertices *A* and A' are $A(a, 0)$ and $A'(-a, 0)$ respectively.

$\therefore \quad AG.A'G = \left(-a + \left(\frac{a^2+b^2}{a}\right)\sec\phi\right)\left(a + \left(\frac{a^2+b^2}{a}\right)\sec\phi\right) = \left(\frac{a^2+b^2}{a}\right)^2 \sec^2\phi - a^2 = (ae^2)^2\sec^2\phi - a^2 = a^2(e^4\sec^2\phi - 1)$

Example: 21 The normal at *P* to a hyperbola of eccentricity *e*, intersects its transverse and conjugate axis at *L* and *M* respectively, then the locus of the middle point of *LM* is a hyperbola whose eccentricity is

(a) $\frac{e}{\sqrt{e^2-1}}$ (b) $\frac{e}{\sqrt{e^4-1}}$ (c) $\frac{e}{\sqrt{a^2e^2-1}}$ (d) None of these

Solution: (a) The equation of the normal at $P(a\sec\phi, b\tan\phi)$ to the hyperbola is $ax\cos\phi + by\cot\phi = a^2 + b^2 = a^2e^2$

It meets the transverse and conjugate axes at *L* and *M*, then $L(ae^2\sec\phi, 0)$; $M\left(0, \frac{a^2e^2\tan\phi}{b}\right)$

Let the middle point of *LM* is (α, β); then $\alpha = \frac{ae^2\sec\phi}{2} \Rightarrow \sec\phi = \frac{2\alpha}{ae^2}$ .....(i)

and $\beta = \frac{a^2e^2\tan\phi}{2b} \Rightarrow \tan\phi = \frac{2b\beta}{a^2e^2}$ ......(ii)

$\because \quad 1 = \sec^2\phi - \tan^2\phi$; $1 = \frac{4\alpha^2}{a^2e^4} - \frac{4b^2\beta^2}{a^4e^4}$, $\therefore$ Locus of (α, β) is $\frac{x^2}{\left(\frac{a^2e^4}{4}\right)} - \frac{y^2}{\left(\frac{a^4e^4}{4b^2}\right)} = 1$

It is a hyperbola, let its eccentricity $e_1 = \frac{\sqrt{\left(\frac{a^2e^4}{4} + \frac{a^4e^4}{4b^2}\right)}}{\left(\frac{a^2e^4}{4}\right)} = \sqrt{1 + \frac{a^2}{b^2}} = \sqrt{\frac{a^2+b^2}{b^2}} = \sqrt{\frac{a^2e^2}{a^2(e^2-1)}}$; $\therefore \quad e_1 = \frac{e}{\sqrt{e^2-1}}$.

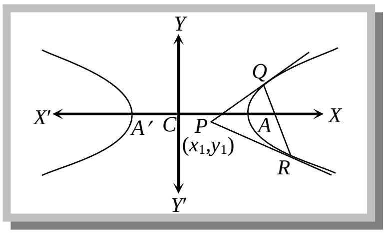
5.3.11 Equation of Chord of Contact of Tangents drawn from a Point to a Hyperbola

Let PQ and PR be tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ drawn from any external point $P(x_1, y_1)$.

Then equation of chord of contact QR is

or $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$

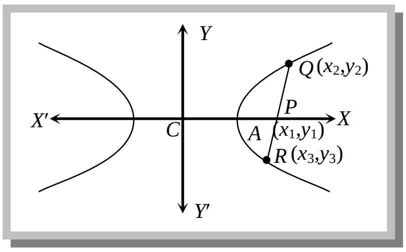
or $T = 0$ (At x_1, y_1)



5.3.12 Equation of the Chord of the Hyperbola whose Mid point (x_1, y_1) is given

Equation of the chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, bisected at the given point (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$

i.e., $T = S_1$



Note : ❑ The length of chord cut off by hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ from the line $y = mx + c$ is

$$\frac{2ab\sqrt{[c^2 - (a^2m^2 - b^2)](1+m^2)}}{(b^2 - a^2m^2)}$$

5.3.13 Equation of the Chord joining Two points on the Hyperbola

The equation of the chord joining the points $P(a\sec\phi_1, b\tan\phi_1)$ and $Q(a\sec\phi_2, b\tan\phi_2)$ is

$$y - b\tan\phi_1 = \frac{b\tan\phi_2 - b\tan\phi_1}{a\sec\phi_2 - a\sec\phi_1}(x - a\sec\phi_1)$$

$$\frac{x}{a}\cos\left(\frac{\phi_1 - \phi_2}{2}\right) - \frac{y}{b}\sin\left(\frac{\phi_1+\phi_2}{2}\right) = \cos\left(\frac{\phi_1+\phi_2}{2}\right)$$

Note : ❑ If the chord joining two points $(a\sec\theta_1, b\tan\theta_1)$ and $(a\sec\theta_2, b\tan\theta_2)$ passes through the focus of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2} = \frac{1-e}{1+e}$.

Example: 22 The equation of the chord of contact of tangents drawn from a point (2, –1) to the hyperbola $16x^2 - 9y^2 = 144$ is

(a) $32x + 9y = 144$ (b) $32x + 9y = 55$ (c) $32x + 9y + 144 = 0$ (d) $32x + 9y + 55 = 0$

Solution: (a) From $T = 0$ *i.e.*, $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$. Here, $16x^2 - 9y^2 = 144$ *i.e.*, $\frac{x^2}{9} - \frac{y^2}{16} = 1$

So, the equation of chord of contact of tangents drawn from a point (2, –1) to the hyperbola is $\frac{2x}{9}-\frac{(-1)y}{16}=1$

i.e., $32x+9y=144$

Example: 23 The point of intersection of tangents drawn to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the points where it is intersected by the line $lx+my+n=0$ is

(a) $\left(\frac{-a^2l}{n},\frac{b^2m}{n}\right)$ (b) $\left(\frac{a^2l}{n},\frac{-b^2m}{n}\right)$ (c) $\left(-\frac{a^2n}{l},\frac{b^2n}{m}\right)$ (d) $\left(\frac{a^2n}{l},\frac{-b^2n}{m}\right)$

Solution: (a) Let (x_1,y_1) be the required point. Then the equation of the chord of contact of tangents drawn from (x_1,y_1) to the given hyperbola is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$ ......(i)

The given line is $lx+my+n=0$ .....(ii)

Equation (i) and (ii) represent the same line

$\therefore\ \frac{x_1}{a^2l}=-\frac{y_1}{b^2m}=\frac{1}{-h}\Rightarrow x_1=\frac{-a^2l}{n},y_1=\frac{b^2m}{n}$; Hence the required point is $\left(-\frac{a^2l}{n},\frac{b^2m}{n}\right)$.

Example: 24 What will be equation of that chord of hyperbola $25x^2-16y^2=400$, whose mid point is (5, 3) **[UPSEAT 1999]**

(a) $115x-117y=17$ (b) $125x-48y=481$ (c) $127x+33y=341$ (d) $15x+121y=105$

Solution: (b) According to question, $S=25x^2-16y^2-400=0$

Equation of required chord is $S_1=T$ ......(i)

Here $S_1=25(5)^2-16(3)^2-400=625-144-400=81$ and $T=25xx_1-16yy_1-400$, where $x_1=5$, $y_1=3$

$\Rightarrow 25x(5)-16y(3)-400=125x-48y-400$

So, from (i) required chord is $125x-48y-400=81\Rightarrow 125x-48y=481$.

Example: 25 The locus of the mid-points of the chords of the circle $x^2+y^2=16$ which are tangent to the hyperbola $9x^2-16y^2=144$ is

(a) $(x^2+y^2)^2=16x^2-9y^2$ (b) $(x^2+y^2)^2=9x^2-16y^2$

(c) $(x^2-y^2)^2=16x^2-9y^2$ (d) None of these

Solution: (a) The given hyperbola is $\frac{x^2}{16}-\frac{y^2}{9}=1$ ......(i)

Any tangent to (i) is $y=mx+\sqrt{16m^2-9}$ ......(ii)

Let (x_1,y_1) be the mid point of the chord of the circle $x^2+y^2=16$

Then equation of the chord is $T=S_1$ *i.e.*, $xx_1+yy_1-(x_1^2+y_1^2)=0$ ......(iii)

Since (ii) and (iii) represent the same line.

$\therefore\ \frac{m}{x_1}=\frac{-1}{y_1}=\frac{\sqrt{16m^2-9}}{-(x_1^2+y_1^2)}$

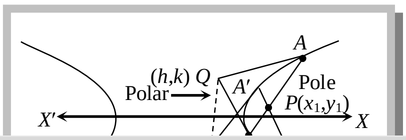
$\Rightarrow\ m=-\frac{x_1}{y_1}$ and $(x_1^2+y_1^2)^2=y_1^2(16m^2-9)\Rightarrow(x_1^2+y_1^2)^2=16.\frac{x_1^2}{y_1^2}y_1^2-9y_1^2=16x_1^2-9y_1^2$

$\therefore$ Locus of (x_1,y_1) is $(x^2+y^2)^2=16x^2-9y^2$.

5.3.14 Pole and Polar

Let P be any point inside or outside the hyperbola. If any straight line drawn through P interesects the hyperbola at A and B. Then the locus of the point of intersection of the tangents to the hyperbola at A and B is called the polar of the given point P with respect to the hyperbola and the point P is called the pole of the polar.

The equation of the required polar with (x_1,y_1) as its pole is



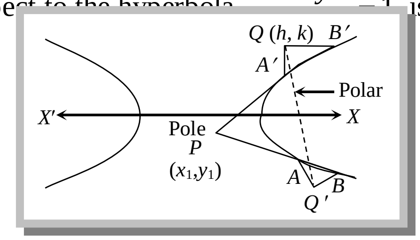
$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

Note : ❑ Polar of the focus is the directrix.

❑ Any tangent is the polar of its point of contact.

(1) **Pole of a given line :** The pole of a given line $lx + my + n = 0$ with respect to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$(x_1, y_1) = \left(-\frac{a^2 l}{n}, \frac{b^2 m}{n}\right)$$



(2) **Properties of pole and polar**

(i) If the polar of $P(x_1, y_1)$ passes through $Q(x_2, y_2)$, then the polar of $Q(x_2, y_2)$ goes through $P(x_1, y_1)$ and such points are said to be conjugate points.

(ii) If the pole of a line $lx + my + n = 0$ lies on the another line $l'x + m'y + n' = 0$ then the pole of the second line will lie on the first and such lines are said to be conjugate lines.

(iii) Pole of a given line is same as point of intersection of tangents as its extremities.

Important Tips

☞ *If the polars of* (x_1, y_1) *and* (x_2, y_2) *with respect to the hyperbola* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ *are at right angles, then* $\frac{x_1 x_2}{y_1 y_2} + \frac{a^4}{b^4} = 0$

Example: 26 If the polar of a point *w.r.t.* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then the locus of the point is **[Pb. CET 1999]**

(a) Given hyperbola (b) Ellipse

(c) Circle (d) None of these

Solution: (a) Let (x_1, y_1) be the given point.

Its polar *w.r.t.* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ *i.e.,* $y = \frac{b^2}{y_1}\left(1 - \frac{xx_1}{a^2}\right) = -\frac{b^2 x_1}{a^2 y_1} x + \frac{b^2}{y_1}$

This touches $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $\left(\frac{b^2}{y_1}\right)^2 = a^2 \cdot \left(\frac{b^2 x_1}{a^2 y_1}\right) - b^2 \Rightarrow \frac{b^4}{y_1^2} = \frac{a^2 b^4 x_1^2}{a^4 y_1^2} - b^2 \Rightarrow \frac{b^2}{y_1^2} = \frac{b^2 x_1^2}{a^2 y_1^2} - 1 \Rightarrow \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$

$\therefore$ Locus of (x_1, y_1) is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Which is the same hyperbola.

Example: 27 The locus of the poles of the chords of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, which subtend a right angle at the centre is

(a) $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2} - \frac{1}{b^2}$ (b) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{1}{a^2} - \frac{1}{b^2}$ (c) $\frac{x^2}{a^4} - \frac{y^2}{b^4} = \frac{1}{a^2} + \frac{1}{b^2}$ (d) $\frac{x^2}{a^4} - \frac{y^2}{b^4} = \frac{1}{a^2} - \frac{1}{b^2}$

Solution: (a) Let (x_1, y_1) be the pole w.r.t. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ......(i)

Then equation of polar is $\frac{hx}{a^2} - \frac{ky}{b^2} = 1$ .....(ii)

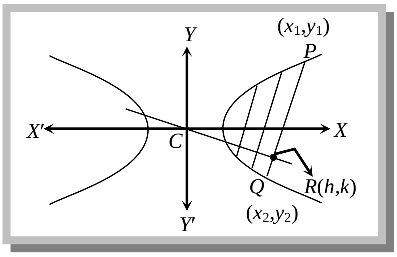
The equation of lines joining the origin to the points of intersection of (i) and (ii) is obtained by making homogeneous (i) with the help of (ii), then $\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)=\left(\frac{hx}{a^2}-\frac{ky}{b^2}\right)^2 \Rightarrow x^2\left(\frac{1}{a^2}-\frac{h^2}{a^4}\right)-y^2\left(\frac{1}{b^2}+\frac{k^2}{b^4}\right)+\frac{2hk}{a^2b^2}xy=0$

Since the lines are perpendicular, then coefficient of x^2 + coefficient of $y^2=0$

$\frac{1}{a^2}-\frac{h^2}{a^4}-\frac{1}{b^2}-\frac{k^2}{b^4}=0$ or $\frac{h^2}{a^4}+\frac{k^2}{b^4}=\frac{1}{a^2}-\frac{1}{b^2}$. Hence required locus is $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2}-\frac{1}{b^2}$

5.3.15 Diameter of the Hyperbola

The locus of the middle points of a system of parallel chords of a hyperbola is called a diameter and the point where the diameter intersects the hyperbola is called the vertex of the diameter.



Let $y=mx+c$ a system of parallel chords to $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ for different chords then the equation of diameter of the hyperbola is $y=\frac{b^2x}{a^2m}$, which is passing through (0, 0)

Conjugate diameter : Two diameters are said to be conjugate when each bisects all chords parallel to the others.

If $y=m_1x$, $y=m_2x$ be conjugate diameters, then $m_1m_2=\frac{b^2}{a^2}$.

Note :
- If a pair of diameters be conjugate with respect to a hyperbola, they are conjugate with respect to its conjugate hyperbola also.
- In a pair of conjugate diameters of a hyperbola. Only one meets the curve in real points.
- The condition for the lines $AX^2+2HXY+BY^2=0$ to be conjugate diameters of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $a^2A=b^2B$.

Important Tips

- *If CD is the conjugate diameter of a diameter CP of the hyperbola* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, *where P is* $(a\sec\phi, b\tan\phi)$ *then coordinates of D is* $(a\tan\phi, b\sec\phi)$, *where C is (0, 0).*

Example: 28 If a pair of conjugate diameters meet the hyperbola and its conjugate in P and D respectively, then $CP^2-CD^2=$

(a) a^2+b^2 (b) a^2-b^2 (c) $\frac{a^2}{b^2}$ (d) None of these

Solution: (b) Coordinates of P and D are $(a\sec\phi, b\tan\phi)$ and $(a\tan\phi, b\sec\phi)$ respectively.

Then $(CP)^2-(CD)^2 = a^2\sec^2\phi+b^2\tan^2\phi-a^2\tan^2\phi-b^2\sec^2\phi$

$= a^2(\sec^2\phi-\tan^2\phi)-b^2(\sec^2\phi-\tan^2\phi) = a^2(1)-b^2(1) = a^2-b^2$.

Example: 29 If the line $lx+my+n=0$ passes through the extremities of a pair of conjugate diameters of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ then

(a) $a^2l^2-b^2m^2=0$ (b) $a^2l^2+b^2m^2=0$ (c) $a^2l^2+b^2m^2=n^2$ (d) None of these

Solution: (a) The extremities of a pair of conjugate diameters of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are $(a\sec\phi, b\tan\phi)$ and $(a\tan\phi, b\sec\phi)$ respectively.

According to the question, since extremities of a pair of conjugate diameters lie on $lx+my+n=0$

$\therefore \quad l(a\sec\phi)+m(b\tan\phi)+n=0 \Rightarrow l(a\tan\phi)+m(b\sec\phi)+n=0$ ......(i)

Then from (i), $al\sec\phi+bm\tan\phi=-n$ or $a^2l^2\sec^2\phi+b^2m^2\tan^2\phi+2ablm\sec\phi\tan\phi=n^2$ ......(ii)

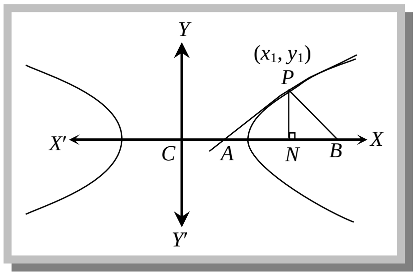
And from (ii), $al\tan\phi+bm\sec\phi=-n$ or $a^2l^2\tan^2\phi+b^2m^2\sec^2\phi+2ablm\sec\phi\tan\phi=n^2$ ......(iii)

Then subtracting (ii) from (iii)

$\therefore \quad a^2l^2(\sec^2\phi-\tan^2\phi)+b^2m^2(\tan^2\phi-\sec^2\phi)=0$ or $a^2l^2-b^2m^2=0$.

5.3.16 Subtangent and Subnormal of the Hyperbola

Let the tangent and normal at $P(x_1, y_1)$ meet the x-axis at A and B respectively.



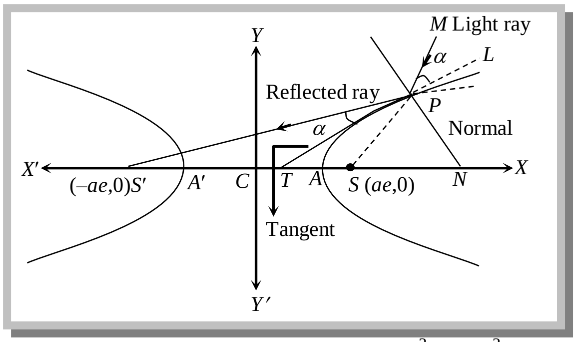
Length of subtangent $AN=CN-CA=x_1-\dfrac{a^2}{x_1}$

Length of subnormal $BN=CB-CN=\dfrac{(a^2+b^2)}{a^2}x_1-x_1=\dfrac{b^2}{a^2}x_1=(e^2-1)x_1$

5.3.17 Reflection property of the Hyperbola

If an incoming light ray passing through one focus (S) strike convex side of the hyperbola then it will get reflected towards other focus (S')

$\angle TPS'=\angle LPM=\alpha$



Example: 30 A ray emanating from the point (5, 0) is incident on the hyperbola $9x^2-16y^2=144$ at the point P with abscissa 8; then the equation of reflected ray after first reflection is (Point P lies in first quadrant)

(a) $3\sqrt{3}x-13y+15\sqrt{3}=0$ (b) $3x-13y+15=0$ (c) $3\sqrt{3}x+13y-15\sqrt{3}=0$ (d) None of these

Solution: (a) Given hyperbola is $9x^2-26y^2=144$. This equation can be rewritten as $\dfrac{x^2}{16}-\dfrac{y^2}{9}=1$ .....(i)

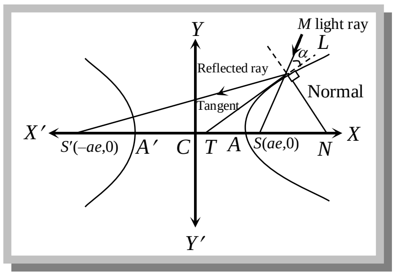
Since x coordinate of P is 8. Let y-coordinate of P is α

$\therefore \quad (8,\alpha)$ lies on (i)

$\therefore \quad \dfrac{64}{16}-\dfrac{\alpha^2}{9}=1$; $\therefore \quad \alpha=27$ $(\because$ P lies in first quadrant)

$\alpha=3\sqrt{3}$

Hence coordinate of point P is $(8, 3\sqrt{3})$



$\because$ Equation of reflected ray passing through $P(8, 3\sqrt{3})$ and $S'(-5, 0)$; $\therefore$ Its equation is $y-3\sqrt{3}=\dfrac{0-3\sqrt{3}}{-5-8}(x-8)$

or $13y-39\sqrt{3}=3\sqrt{3}x-24\sqrt{3}$ or $3\sqrt{3}x-13y+15\sqrt{3}=0$

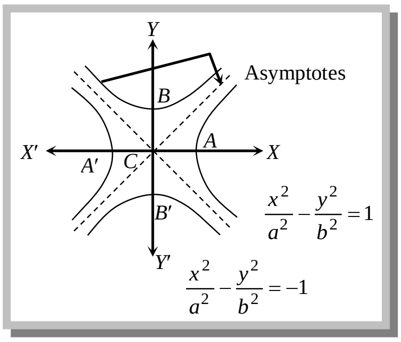
5.3.18 Asymptotes of a Hyperbola

An asymptote to a curve is a straight line, at a finite distance from the origin, to which the tangent to a curve tends as the point of contact goes to infinity.

The equations of two asymptotes of the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ are $y=\pm\dfrac{b}{a}x$ or $\dfrac{x}{a}\pm\dfrac{y}{b}=0$.

Note :

- The combined equation of the asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$.
- When $b=a$ *i.e.* the asymptotes of rectangular hyperbola $x^2-y^2=a^2$ are $y=\pm x$, which are at right angles.
- A hyperbola and its conjugate hyperbola have the same asymptotes.
- The equation of the pair of asymptotes differ the hyperbola and the conjugate hyperbola by the same constant only *i.e.* Hyperbola – Asymptotes = Asymptotes – Conjugated hyperbola or, $\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1\right)-\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)=\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)-\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}+1\right)$.

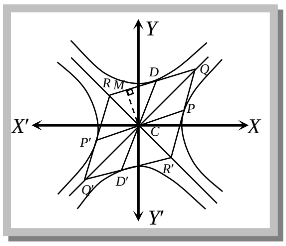


- The asymptotes pass through the centre of the hyperbola.
- The bisectors of the angles between the asymptotes are the coordinate axes.
- The angle between the asymptotes of the hyperbola $S=0$ *i.e.,* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $2\tan^{-1}\frac{b}{a}$ or $2\sec^{-1}e$.
- Asymptotes are equally inclined to the axes of the hyperbola.

Important Tips

☞ *The parallelogram formed by the tangents at the extremities of conjugate diameters of a hyperbola has its vertices lying on the asymptotes and is of constant area.*

Area of parallelogram $QRQ'R'$ = *4(Area of parallelogram QDCP)* = $4ab$ = *Constant*



☞ *The product of length of perpendiculars drawn from any point on the hyperbola* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ *to the asymptotes is* $\frac{a^2b^2}{a^2+b^2}$.

Example: 31 From any point on the hyperbola, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ tangents are drawn to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=2$. The area cut-off by the chord of contact on the asymptotes is equal to

(a) $\frac{ab}{2}$ (b) ab (c) $2ab$ (d) $4ab$

Solution: (d) Let $P(x_1, y_1)$ be a point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then $\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=1$

The chord of contact of tangent from P to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=2$ is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=2$ .....(i)

The equation of asymptotes are $\frac{x}{a}-\frac{y}{b}=0$ ......(ii)

And $\frac{x}{a}+\frac{y}{b}=0$ ......(iii)

The point of intersection of the asymptotes and chord are $\left(\frac{2a}{x_1/a-y_1/b},\frac{2b}{x_1/a-y_1/b}\right);\left(\frac{2a}{x_1/a+y_1/b},\frac{-2b}{x_1/a+y_1/b}\right)$, (0, 0)

$\therefore$ Area of triangle $= \frac{1}{2}|(x_1y_2 - x_2y_1)| = \frac{1}{2}\left|\left(\frac{-8ab}{x_1^2/a^2 - y_1^2/b^2}\right)\right| = 4ab$.

Example: 32 The combined equation of the asymptotes of the hyperbola $2x^2+5xy+2y^2+4x+5y=0$ **[Karnataka CET 2002]**

(a) $2x^2+5xy+2y^2=0$ (b) $2x^2+5xy+2y^2-4x+5y+2=0=0$

(c) $2x^2+5xy+2y^2+4x+5y-2=0$ (d) $2x^2+5xy+2y^2+4x+5y+2=0$

Solution: (d) Given, equation of hyperbola $2x^2+5xy+2y^2+4x+5y=0$ and equation of asymptotes

$2x^2+5xy+2y^2+4x+5y+\lambda=0$ ......(i) which is the equation of a pair of straight lines. We know that the standard equation of a pair of straight lines is $ax^2+2hxy+by^2+2gx+2fy+c=0$

Comparing equation (i) with standard equation, we get $a=2, b=2$, $h=\frac{5}{2}, g=2, f=\frac{5}{2}$ and $c=\lambda$.

We also know that the condition for a pair of straight lines is $abc+2fgh-af^2-bg^2-ch^2=0$.

Therefore, $4\lambda+25-\frac{25}{2}-8-\frac{25}{4}\lambda=0$ or $\frac{-9\lambda}{4}+\frac{9}{2}=0$ or $\lambda=2$

Substituting value of λ in equation (i), we get $2x^2+5xy+2y^2+4x+5y+2=0$.

5.3.19 Rectangular or Equilateral Hyperbola

(1) **Definition :** A hyperbola whose asymptotes are at right angles to each other is called a rectangular hyperbola. The eccentricity of rectangular hyperbola is always $\sqrt{2}$.

The general equation of second degree represents a rectangular hyperbola if $\Delta \neq 0$, $h^2 > ab$ and coefficient of x^2 + coefficient of $y^2 = 0$

The equation of the asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are given by $y=\pm\frac{b}{a}x$.

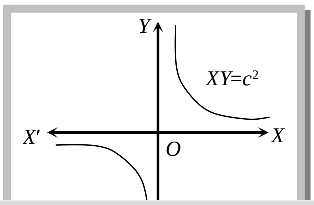
The angle between these two asymptotes is given by $\tan\theta = \frac{\frac{b}{a}-\left(-\frac{b}{a}\right)}{1+\frac{b}{a}\left(\frac{-b}{a}\right)} = \frac{2b/a}{1-b^2/a^2} = \frac{2ab}{a^2-b^2}$.

If the asymptotes are at right angles, then $\theta=\pi/2 \Rightarrow \tan\theta=\tan\frac{\pi}{2} \Rightarrow \frac{2ab}{a^2-b^2}=\tan\frac{\pi}{2} \Rightarrow a^2-b^2=0$ $\Rightarrow a=b \Rightarrow 2a=2b$. Thus the transverse and conjugate axis of a rectangular hyperbola are equal and the equation is $x^2-y^2=a^2$. The equations of the asymptotes of the rectangular hyperbola are $y=\pm x$ *i.e.*, $y=x$ and $y=-x$. Clearly, each of these two asymptotes is inclined at 45° to the transverse axis.

(2) **Equation of the rectangular hyperbola referred to its asymptotes as the axes of coordinates :** Referred to the transverse and conjugate axis as the axes of coordinates, the equation of the rectangular hyperbola is

$$x^2-y^2=a^2 \qquad .....(i)$$

The asymptotes of (i) are $y = x$ and $y = -x$. Each of these two asymptotes is inclined at an angle of 45° with the transverse axis, So, if we rotate the coordinate axes through an angle of $-\pi/4$ keeping the origin fixed, then the axes coincide with the asymptotes of the hyperbola and $x = X\cos(-\pi/4) - Y\sin(-\pi/4) = \frac{X+Y}{\sqrt{2}}$ and $y = X\sin(-\pi/4) + Y\cos(-\pi/4) = \frac{Y-X}{\sqrt{2}}$.



Substituting the values of x and y in (i), We obtain the $\left(\frac{X+Y}{\sqrt{2}}\right)^2 - \left(\frac{Y-X}{\sqrt{2}}\right)^2 = a^2 \Rightarrow XY = \frac{a^2}{2} \Rightarrow XY = c^2$

where $c^2 = \frac{a^2}{2}$.

This is transformed equation of the rectangular hyperbola (i).

(3) **Parametric co-ordinates of a point on the hyperbola $XY = c^2$:** If t is non–zero variable, the coordinates of any point on the rectangular hyperbola $xy = c^2$ can be written as $(ct, c/t)$. The point $(ct, c/t)$ on the hyperbola $xy = c^2$ is generally referred as the point 't'.

For rectangular hyperbola the coordinates of foci are $(\pm a\sqrt{2}, 0)$ and directrices are $x = \pm a\sqrt{2}$.

For rectangular hyperbola $xy = c^2$, the coordinates of foci are $(\pm c\sqrt{2}, \pm c\sqrt{2})$ and directrices are $x + y = \pm c\sqrt{2}$.

(4) **Equation of the chord joining points t_1 and t_2 :** The equation of the chord joining two points $\left(ct_1, \frac{c}{t_1}\right)$ and $\left(ct_2, \frac{c}{t_2}\right)$ on the hyperbola $xy = c^2$ is $y - \frac{c}{t_1} = \frac{\frac{c}{t_2} - \frac{c}{t_1}}{ct_2 - ct_1}(x - ct_1) \Rightarrow x + y\,t_1 t_2 = c(t_1 + t_2)$.

(5) **Equation of tangent in different forms**

(i) **Point form :** The equation of tangent at (x_1, y_1) to the hyperbola $xy = c^2$ is $xy_1 + yx_1 = 2c^2$ or $\frac{x}{x_1} + \frac{y}{y_1} = 2$

(ii) **Parametric form :** The equation of the tangent at $\left(ct, \frac{c}{t}\right)$ to the hyperbola $xy = c^2$ is $\frac{x}{t} + yt = 2c$ .On replacing x_1 by ct and y_1 by $\frac{c}{t}$ on the equation of the tangent at (x_1, y_1) i.e. $xy_1 + yx_1 = 2c^2$ we get $\frac{x}{t} + yt = 2c$.

Note : ☐ Point of intersection of tangents at 't_1' and 't_2' is $\left(\frac{2ct_1t_2}{t_1 + t_2}, \frac{2c}{t_1 + t_2}\right)$

(6) **Equation of the normal in different forms :** (i) **Point form :** The equation of the normal at (x_1, y_1) to the hyperbola $xy = c^2$ is $xx_1 - yy_1 = x_1^2 - y_1^2$. As discussed in the equation of the tangent, we have $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\frac{y_1}{x_1}$

So, the equation of the normal at (x_1, y_1) is $y - y_1 = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}(x - x_1) \Rightarrow y - y_1 = \frac{x_1}{y_1}(x - x_1)$

$\Rightarrow yy_1 - y_1^2 = xx_1 - x_1^2 \quad \Rightarrow xx_1 - yy_1 = x_1^2 - y_1^2$

This is the required equation of the normal at (x_1, y_1).

(ii) **Parametric form:** The equation of the normal at $\left(ct, \frac{c}{t}\right)$ to the hyperbola $xy = c^2$ is $xt^3 - yt - ct^4 + c = 0$.

On replacing x_1 by ct and y_1 by c/t in the equation.

We obtain $xx_1 - yy_1 = x_1^2 - y_1^2, \quad xct - \frac{yc}{t} = c^2t^2 - \frac{c^2}{t^2} \Rightarrow xt^3 - yt - ct^4 + c = 0$

Note : ☐ The equation of the normal at $\left(ct, \frac{c}{t}\right)$ is a fourth degree in t. So, in general, four normals can be drawn from a point to the hyperbola $xy = c^2$

- If the normal at $\left(ct, \frac{c}{t}\right)$ on the curve $xy = c^2$ meets the curve again in 't'' then; $t' = \frac{-1}{t^3}$.
- Point of intersection of normals at 't_1' and 't_2' is $\left(\frac{c\{t_1t_2(t_1^2 + t_1t_2 + t_2^2) - 1\}}{t_1t_2(t_1 + t_2)}, \frac{c\{t_1^3t_2^3 + (t_1^2 + t_1t_2 + t_2^2)\}}{t_1t_2(t_1 + t_2)}\right)$

Important Tips

- *A triangle has its vertices on a rectangular hyperbola; then the orthocentre of the triangle also lies on the same hyperbola.*
- *All conics passing through the intersection of two rectangular hyperbolas are themselves rectangular hyperbolas.*
- *An infinite number of triangles can be inscribed in the rectangular hyperbola* $xy = c^2$ *whose all sides touch the parabola* $y^2 = 4ax$.

Example: 33 If $5x^2 + \lambda y^2 = 20$ represents a rectangular hyperbola, then λ equals

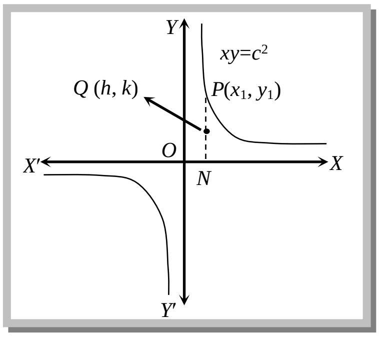
(a) 5 (b) 4 (c) – 5 (d) None of these

Solution: (c) Since the general equation of second degree represents a rectagular hyperbola if $\Delta \neq 0, h^2 > ab$ and coefficient of x^2 + coefficient of $y^2 = 0$. Therefore the given equation represents a rectangular hyperbola if $\lambda + 5 = 0$ *i.e.,* $\lambda = -5$

Example: 34 If *PN* is the perpendicular from a point on a rectangular hyperbola to its asymptotes, the locus, the mid-point of *PN* is

(a) Circle (b) Parabola (c) Ellipse (d) Hyperbola

Solution: (d) Let $xy = c^2$ be the rectangular hyperbola, and let $P(x_1, y_1)$ be a point on it. Let $Q(h,k)$ be the mid-point of *PN*. Then the coordinates of *Q* are $\left(x_1, \frac{y_1}{2}\right)$.



$\therefore$ $x_1 = h$ and $\frac{y_1}{2} = k \Rightarrow x_1 = h$ and $y_1 = 2k$

But (x_1, y_1) lies on $xy = c^2$.

$\therefore$ $h.(2k) = c^2 \Rightarrow hk \Rightarrow c^2/2$

Hence, the locus of (h,k) is $xy = c^2/2$, which is a hyperbola.

Example: 35 If the normal at $\left(ct, \frac{c}{t}\right)$ on the curve $xy = c^2$ meets the curve again in t', then

(a) $t' = -\frac{1}{t^3}$ (b) $t' = -\frac{1}{t}$ (c) $t' = \frac{1}{t^2}$ (d) $t'^2 = -\frac{1}{t^2}$

Solution: (a) The equation of the tangent at $\left(ct, \frac{c}{t}\right)$ is $ty = t^3x - ct^4 + c$

If it passes through $\left(ct', \frac{c}{t'}\right)$ then

$$\Rightarrow \frac{tc}{t'} = t^3ct' - ct^4 + c \Rightarrow t = t^3t'^2 - t^4t' + t' \Rightarrow t - t' = t^3t'(t' - t) \Rightarrow t' = -\frac{1}{t^3}$$

Example: 36 If the tangent and normal to a rectangular hyperbola cut off intercepts a_1 and a_2 on one axis and b_1 and b_2 on the other axis, then

(a) $a_1b_1 + a_2b_2 = 0$ (b) $a_1b_2 + b_2a_1 = 0$ (c) $a_1a_2 + b_1b_2 = 0$ (d) None of these

Solution: (c) Let the hyperbola be $xy = c^2$. Tangent at any point *t* is $x + yt^2 - 2ct = 0$

Putting $y = 0$ and then $x = 0$ intercepts on the axes are $a_1 = 2ct$ and $b_1 = \frac{2c}{t}$

Normal is $xt^3 - yt - ct^4 + c = 0$.

Intercepts as above are $a_2 = \frac{c(t^4 - 1)}{t^3}$, $b^2 = \frac{-c(t^4 - 1)}{t}$

$\therefore \quad a_1a_2 + b_1b_2 = 2ct \times \frac{c(t^4-1)}{t^3} + \frac{2c}{t} \times \frac{-c(t^4-1)}{t} = \frac{2c^2}{t^2}(t^4-1) - \frac{2c^2}{t^2}(t^4-1) = 0$; $\therefore \quad a_1a_2 + b_1b_2 = 0$.

Example: 37 A variable straight line of slope 4 intersects the hyperbola $xy = 1$ at two points. The locus of the point which divides the line segment between these two points in the ratio 1 : 2 is **[IIT 1997]**

(a) $16x^2 + 10xy + y^2 = 2$ (b) $16x^2 - 10xy + y^2 = 2$ (c) $16x^2 + 10xy + y^2 = 4$ (d) None of these

Solution: (a) Let $P(h, k)$ be any point on the locus. Equation of the line through P and having slope 4 is $y - k = 4(x - h)$ .....(i)

Suppose this meets $xy = 1$ ......(ii) in $A(x_1, y_1)$ and $B(x_2, y_2)$

Eliminating y between (i) and (ii), we get $\frac{1}{x} - k = 4(x - h)$

$\Rightarrow \quad 1 - xk = 4x^2 - 4hx \quad \Rightarrow \quad 4x^2 - (4h - k)x - 1 = 0$ ......(iii)

This has two roots say x_1, x_2; $x_1 + x_2 = \frac{4h - k}{4}$ ......(iv) and $x_1x_2 = -\frac{1}{4}$ ......(v)

Also, $\frac{2x_1 + x_2}{3} = h$ [$\because$ P divides AB in the ratio 1 : 2]

i.e., $2x_1 + x_2 = 3h$ ......(vi)

(vi) – (iv) gives, $x_1 = 3h - \frac{4h-k}{4} = \frac{8h+k}{4}$ and $x_2 = 3h - 2.\frac{8h+k}{4} = -\frac{2h+k}{2}$

Putting in (v), we get $\frac{8h+k}{4}\left(-\frac{2h+k}{2}\right) = -\frac{1}{4}$

$\Rightarrow \quad (8h + k)(2h + k) = 2 \quad \Rightarrow \quad 16h^2 + 10hk + k^2 = 2$

$\therefore$ Required locus of $P(h, k)$ is $16x^2 + 10xy + y^2 = 2$.

Example: 38 PQ and RS are two perpendicular chords of the rectangular hyperbola $xy = c^2$. If C is the centre of the rectangular hyperbola, then the product of the slopes of CP, CQ, CR and CS is equal to

(a) – 1 (b) 1 (c) 0 (d) None of these

Solution: (b) Let t_1, t_2, t_3, t_4 be the parameters of the points P, Q, R and S respectively. Then, the coordinates of P, Q, R and S are $\left(ct_1, \frac{c}{t_1}\right)$, $\left(ct_2, \frac{c}{t_2}\right)$, $\left(ct_3, \frac{c}{t_3}\right)$ and $\left(ct_4, \frac{c}{t_4}\right)$ respectively.

Now, $PQ \perp RS \quad \Rightarrow \frac{\frac{c}{t_2} - \frac{c}{t_1}}{ct_2 - ct_1} \times \frac{\frac{c}{t_4} - \frac{c}{t_3}}{ct_4 - ct_3} = -1 \Rightarrow -\frac{1}{t_1t_2} \times -\frac{1}{t_3t_4} = -1 \quad \Rightarrow t_1t_2t_3t_4 = -1$ .....(i)

$\therefore$ Product of the slopes of CP, CQ, CR and CS

$\frac{1}{t_1^2} \times \frac{1}{t_2^2} \times \frac{1}{t_3^2} \times \frac{1}{t_4^2} = \frac{1}{t_1^2t_2^2t_3^2t_4^2} = 1$ [Using (i)]

5.3.20 Intersection of a Circle and a Rectangular Hyperbola

If a circle $x^2 + y^2 + 2gx + 2fy + k = 0$ cuts a rectangular hyperbola $xy = c^2$ in A, B, C and D and the parameters of these four points be t_1, t_2, t_3 and t_4 respectively; then

(1) (i) $\Sigma t_1 = -\frac{2g}{c}$ (ii) $\Sigma t_1t_2 = \frac{k}{c^2}$

(iii) $\Sigma t_1t_2t_3 = \frac{-2f}{c}$ (iv) $t_1t_2t_3t_4 = 1$ (v) $\Sigma \frac{1}{t_1} = -\frac{2f}{c}$

(2) Orthocentre of ΔABC is $H\left(-ct_4, \frac{-c}{t_4}\right)$ but D is $\left(ct_4, \frac{c}{t_4}\right)$

Hence H and D are the extremities of a diagonal of rectangular hyperbola.

(3) Centre of mean position of four points is $\left\{\frac{c}{4}\Sigma t_1, \frac{c}{4}\Sigma\left(\frac{1}{t_1}\right)\right\}$ *i.e.*, $\left(-\frac{g}{2}, -\frac{f}{2}\right)$

$\because$ Centres of the circles and rectangular hyperbola are (– g, – f) and (0, 0); mid point of centres of circle and hyperbola is $\left(-\frac{g}{2}, -\frac{f}{2}\right)$. Hence the centre of the mean position of the four points bisects the distance between the centres of the two curves (circle and rectangular hyperbola)

(4) If the circle passing through ABC meet the hyperbola in fourth points D; then centre of circle is (–g, –f)

i.e., $\left\{\frac{c}{2}\left(t_1 + t_2 + t_3 + \frac{1}{t_1 t_2 t_3}\right); \frac{c}{2}\left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + t_1 t_2 t_3\right)\right\}$

Example: 39 If a circle cuts a rectangular hyperbola $xy = c^2$ in A, B, C, D and the parameters of these four points be t_1, t_2, t_3 and t_4 respectively. Then **[Kurukshetra CEE 1998]**

(a) $t_1 t_2 = t_3 t_4$ (b) $t_1 t_2 t_3 t_4 = 1$ (c) $t_1 = t_2$ (d) $t_3 = t_4$

Solution: (b) Let the equation of circle be $x^2 + y^2 = a^2$ ......(i)

Parametric equation of rectangular hyperbola is $x = c\,t, y = \frac{c}{t}$

Put the values of x and y in equation (i) we get $c^2t^2 + \frac{c^2}{t^2} = a^2 \Rightarrow c^2t^4 - a^2t^2 + c^2 = 0$

Hence product of roots $t_1 t_2 t_3 t_4 = \frac{c^2}{c^2} = 1$

Example: 40 If the circle $x^2 + y^2 = a^2$ intersects the hyperbola $xy = c^2$ in four points $P(x_1, y_1), Q(x_2, y_2), R(x_3, y_3)$, $S(x_4, y_4)$ then **[IIT 1998]**

(a) $x_1 + x_2 + x_3 + x_4 = 0$ (b) $y_1 + y_2 + y_3 + y_4 = 0$ (c) $x_1 x_2 x_3 x_4 = c^4$ (d) $y_1 y_2 y_3 y_4 = c^4$

Solution: (a,b,c,d) Given, circle is $x^2 + y^2 = a^2$ ....(i) and hyperbola be $xy = c^2$ .....(ii)

from (ii) $y = \frac{c^2}{x}$. Putting in (i), we get $x^2 + \frac{c^4}{x^2} = a^2 \Rightarrow x^4 - a^2x^2 + c^4 = 0$

$\therefore$ $x_1 + x_2 + x_3 + x_4 = 0$, $x_1 x_2 x_3 x_4 = c^4$

Since both the curves are symmetric in x and y, $\therefore$ $y_1 + y_2 + y_3 + y_4 = 0$; $y_1 y_2 y_3 y_4 = c^4$.

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